SPRING 2023: MATH 590 HOMEWORK

The page and section numbers in the assignments below refer to those in the course textbook.

Wednesday, January 18. Section 1.1: 3, 5, 6, 7, 12.

Friday, January 20. Section 1.2: 3, 13, 15 and Section 1.3: 1a, 1c, 3, 11a and the following problem. Use facts from a first course in linear algebra to prove that if $W \subsetneq \mathbb{R}^2$ is a non-zero proper subspace, then W is a line through the origin.

Monday, January 23. Section 1.3: 1, 7, 9, 10, 11b.

Wednesday, January 25. Section 1.4: 1(a) - 1(f), 4, 5, 7, 12a.

Friday, January 27. Section 1.4: 9a, 10a, and the following problem. Determine if the vectors $v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 7 \\ 5 \\ 0 \end{pmatrix}$ are linearly dependent or linearly independent. If they are linearly dependent, provide a non-trivial dependence relation among them.

Monday, January 30. Section 1.6: 2a, 2b, 12.

Wednesday, February 1. Section 1.6: 7, 9b, 15.

Friday, February 3. Section 2.1: 3.

Monday, February 6. Section 2.2: 7, 10, 12, 13.

Wednesday, February 8. Section 2.2: 3, 5, 9 and the following problem. Let V be a vector space with basis $\alpha := \{v_1, \ldots, v_n\}$. If $v = a_1v_1 + \cdots + a_nv_n$ belongs to V, set $[v]_\alpha := \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. For $c, d \in F$ and $v, u \in V$, show that $[cv + du]_\alpha = c[v]_\alpha + d[u]_\alpha$. Note: This problem will appear on Quiz 4.

Friday, February 10. 1. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $T(x, y) = (x - 2y, x + 2y, 3x - 4y)$. Let $E := \{(1, -1), (1, 2)\}$ and $F := \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ be bases for $\mathbb{R}^2$ and $\mathbb{R}^3$. Verify the formula $[T(v)]_F = [T]_E^F \cdot [v]_E$, for $v = (3, 2)$.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (y, x)$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $S(x, y) = (x + 7, x - y)$. Let $\alpha := \{(-1, 1), (0, 1)\}$, $\beta := \{(1, 0), (1, 1)\}$, and $\gamma := \{(0, -1), (-2, 0)\}$ be bases for $\mathbb{R}^2$. Verify that $[ST]_\alpha^\gamma = [S]_\beta^\gamma \cdot [T]_\alpha^\beta$.

Monday, February 13. Section 2.7: 1,2 and for each of these problems, verify the change of basis theorem, using the given bases.

Wednesday, February 15. Section 2.3: 1b, 1c, 1e, 3a, 3d, 5.

Friday, February 17. Section 2.3: 1f, 4 (but only for 1b, 1c), 7a, 7b.

Monday, February 20. Section 2.3: 1, 2. Note a linear transformation $T : V \to W$ is *injective* if it is 1-1. This happens exactly when $\ker(T) = \{\vec{0}\}$ and is *surjective* if it is onto, which happens exactly when $\mathrm{im}(T) = W$. Equivalently, T is injective if its kernel has dimension zero and T is surjective if the dimension of $\mathrm{im}(T)$ equals the dimension of W.

Monday, February 27. Section 3.3: 1(a), 1(b), and the following problem: Calculate the determinants of the matrices given in Section 3.2a: 1(a) and 1(f) in three ways: expanding along the second row, expanding along the third column, using elementary row operations to reduce to an upper triangular matrix.

Wednesday, March 1. Section 3.3: 7a, 7b, 9, 10. For 9 and 10 prove the statements only for 3×3 matrices.

Friday March 3. Chapter 3, Supplementary Exercise 9a, and the following problems:

(i) Verify $|AB| = |A| \cdot |B|$, for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$.

(ii) Find an orthonormal basis consisting of eigenvectors for the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Be sure to check that your basis is orthonormal.

Monday, March 6. Section 4.5: 1, 2, 3d.

Wednesday, March 8. Section 4.5: 7a,b,c,d for 2×2 matrices. Also: Use the definition of a symmetric linear transformation to show that $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + 2y, 2x + y)$ is symmetric.

Friday, March 10. Section 4.1: 1a, 1c, 2a, 2b, 2c.

Monday, March 20. Section 4.1: 3a, 3c, 3f, 4, 5 (for 3×3 matrices).

Wednesday, March 22. Section 4.2: 1a, 1b, 1c, 1d, 3.

Friday, March 24. Section 4.2: 1e, 1f, 6a, 7.

Monday, March 27. Section 4.3: 2, 9a, 10a, 10b.

Wednesday March 29. Let V denote the three dimensional vector space of real polynomials having degree less than or equal to two with inner product $\langle f(x), g(x)\rangle := \int_{-1}^{1} f(x)g(x)\, dx$. Verify that $f_1 := \frac{1}{\sqrt{2}}$, $f_2 := \sqrt{\frac{3}{2}}x$, $f_3 := \frac{3\sqrt{10}}{4}(x^2 - 3)$ is an orthonormal basis for V and then write $p(x) = 1 + x + x^2$ in terms of this basis.

Friday, March 31. Section 4.4: 5a, 6 and the following problem. Let

$$v_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, v_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, v_3 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

be linearly independent vectors in the space of 2×2 real matrices with inner product $\langle A, B\rangle := \text{trace}(A^t B)$. Find an orthonormal basis for $\text{Span}\{v_1, v_2, v_3\}$.

Monday, April 3. Section 4.4: 1, 2, 3.

Monday, April 10. Verify the following properties for the matrix $A := \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$: (i) $\text{rank}(A^t A) = \text{rank}(AA^t)$; (ii) The eigenvalues of $A^t A$ and AA^t are non-negative real numbers; (iii) $A^t A$ and AA^t have the same **non-zero** eigenvalues with the same multiplicities.

Wednesday, April 12. Find the singular value decomposition of $A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ by following the steps below:

1. Calculate $A^t A$ and its characteristic polynomial $p_{A^t A}(x)$.

2. Find the non-zero eigenvalues of $A^t A$: $\lambda_1 > \lambda_2 > 0$.

3. Find: (i) A unit eigenvector u_1 of λ_1, a unit eigenvector u_2 for λ_2 and a unit vector u_3 such that u_1, u_2, u_3 is an orthonormal basis for $\mathbb{R}^3$.

4. Set $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}$, $v_1 = \frac{1}{\sigma_1} A u_1$, and $v_2 = \frac{1}{\sigma_2} A u_2$. Show that v_1, v_2 is an orthonormal basis for $\mathbb{R}^2$.

5. Let P be the orthogonal matrix whose columns are u_1, u_2, u_3, Q the orthogonal matrix whose columns are v_1, v_2, and $\sum = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{pmatrix}$. Verify that $A = Q \sum P^t$.

Friday, April 14. 1. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$. In our April 10 class we found the SVD of A by applying the Spectral Theorem to A^tA. Now find the SVD of A by applying the Spectral Theorem to AA^t.

2. For an $m \times n$ matrix A with SVD $A = Q \sum P^t$, the *pseudo-inverse* of A is the matrix $A^\dagger := P \sum^\dagger Q^t$, where $\sum^\dagger$ is the $n \times m$ diagonal matrix whose non-zero diagonal entries are $\frac{1}{\sigma_1}, \dots, \frac{1}{\sigma_r}$. Find the pseudo-inverse of the matrix A in problem 1 above.

Monday, April 17. 1. For the matrix A in problem 1 from April 14, find a best approximation to the system $A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, which has no solution.

2. For the complex numbers $z_1 = a + bi$, $z_2 = c + di$, verify the following formulas. Here $\overline{z_1} = a - bi$, the complex conjugate of z_1.

(i) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$.
(ii) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.
(iii) $\sqrt{z_1 \cdot z_1} = \sqrt{a^2 + b^2}$. This quantity is denoted $|z_1|$ and is called the absolute value or *modulus* of z_1.
(iv) $z_1 \cdot z_2 = z_2 \cdot z_1$.
(v) $z_1 \cdot z_2 = z_2 \cdot z_1$.
(vi) $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$, for $z_3 = u + vi$.

Wednesday, April 19. Section 5.1: 1; Section 5.2: 5; Section 5.3: 1, 3a.

Friday, April 21. 1. Show that the matrix $A = \begin{pmatrix} 3 & 2i \\ -2i & 3 \end{pmatrix}$ is normal and then find a unitary matrix P such that P^*AP is a diagonal matrix.

2. Find the singular value decomposition for $A = \begin{pmatrix} i & 0 \\ i & i \\ 0 & i \end{pmatrix}$.

Monday, April 24. Let $A = \begin{pmatrix} 0 & 25 \\ 1 & 10 \end{pmatrix}$. Follow the steps below to arrive at the JCF for A.

(i) Find $p_A(x)$ and the eigenvalue λ of A.
(ii) Calculate E_λ.
(iii) Find a vector $v_2 \notin E_\lambda$.
(iv) Set $v_1 := (A - \lambda I)v_2$.
(v) Let P denote the 2×2 matrix with columns v_1, v_2 and find P^{-1}.
(vi) Verify that $P^{-1}AP$ is the JCF of A.

Wednesday, April 26. Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & -1 \\ -6 & -5 & -3 \end{pmatrix}$. Follow the steps below to find the JCF of A and the change of basis matrix P.

(i) Find $p_A(x)$ and the two eigenvalues λ_1, λ_2. Arrange the eigenvalues so that λ_1 is the eigenvalue with algebraic multiplicity 2.
(ii) Calculate E_{λ_1}.
(ii) Find a vector v_2 in the null space of $(A - \lambda_1 I)^2$ that is not in E_{λ_1}.
(iv) Set $v_1 := (A - \lambda_1 I)v_2$.
(v) Take v_3 any eigenvector associated to λ_2.
(vi) Letting P be the matrix whose columns are v_1, v_2, v_3 verify that $P^{-1}AP = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$.

Friday, April 28. 1. Let $A = \begin{pmatrix} 4 & 0 & -2 \\ 1 & 2 & -1 \\ 2 & 0 & 0 \end{pmatrix}$. Follow the steps below to find the JCF of A and the change of basis matrix P.

(i) Find $p_A(x)$ and the single eigenvalue λ.
(ii) Calculate E_λ.
(iii) Find $v_2 \notin E_\lambda$.
(iv) Set $v_1 := (A - \lambda I)v_2$. This turns out to be a vector in E_λ.
(v) Take $v_3 \in E_\lambda$ not a multiple of v_1.
(vi) Letting P be the matrix whose columns are v_1, v_2, v_3, verify that $P^{-1}AP = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$.

2. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$. Follow the steps below to find the JCF of A and the change of basis matrix P.

(i) Find $p_A(x)$ and the single eigenvalue λ.
(ii) Calculate E_λ.
(iii) Calculate $(A - \lambda I)^2$.
(iv) Find v_3 not in the null space of $(A - \lambda I)^2$.
(v) Take $v_2 := (A - \lambda I)v_3$ and $v_1 := (A - \lambda I)v_2$.
(vi) Letting P be the matrix whose v_1, v_2, v_3, verify that $P^{-1}AP = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$.